Suppose $f : [a, b] \to \mathbb{R}$ is integrable. The **Fundamental Theorem of Calculus** states the following:

1. The function $F : [a, b] \to \mathbb{R}$ defined by
$$F(x) = \int_a^x f(t)\,dt$$
is continuous. F is also differentiable wherever f is continuous, with $F'(x) = f(x)$ in this case.

2. Suppose that F is a continuous anti-derivative of f which is differentiable at all but finitely many points. Then
$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Problem 1

Let $F : (0, \infty) \to \mathbb{R}$ be defined as
$$F(x) = \int_x^{x^2} \frac{1}{t} dt$$

1. Determine for which x we have $F(x) \geq 0$ and for which x we have $F(x) < 0$.
2. Find an expression for $F'(x)$ which involves no integral signs.

Problem 2

1. Prove that if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all $x \in [a, b]$ then
$$\int_a^b f(x)\ dx = (b - a)\mu$$
for some number μ with $m \leq \mu \leq M$.

2. Prove that if f is continuous on $[a, b]$, then
$$\int_a^b f(x)\ dx = (b - a)f(c)$$
for some $c \in [a, b]$. Give a counterexample to this when f is not continuous on $[a, b]$.

3. Let f be continuous on $[a, b]$ and g be integrable and nonnegative on $[a, b]$. Prove that
$$\int_a^b f(x)g(x)\ dx = f(c) \int_a^b g(x)\ dx$$
for some $c \in [a, b]$. You may assume that the integral exists.

Problem 3
Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Show that there does not exist a function $F : \mathbb{R} \to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in \mathbb{R}$. *Hint: Recall that if $F'(x) = 0$ along an interval, then F is constant on that interval.*

Also, find $\int f$. This shows that we can't always use the FTC to find the integral, and that defining the integral as the "inverse" of the derivative is too restrictive.

Problem 4
For a fixed $a \in \mathbb{R}$, consider the function $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) = \int_0^a \left(\int_a^x st^2 dt \right) ds$$

What is F'? *Hint: Move outside.*

Recall the integration by parts formula:

$$\int_a^b u(x)v'(x)\, dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)\, dx,$$

or the mnemonic,

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du.$$

Problem 5
Let $I = \int e^x \sin(x) dx$. Apply integration by parts (twice) to obtain an expression for I in terms of itself. Solve for I to obtain an expression with no integral sign.